Real Analysis tutorial class 2

Example 0.1. *Given $f_n \in C^0(\mathbb{R})$, then the set where f_n converges is a intersection of F_σ sets.*

Proof. f_n is convergent at $a \in \mathbb{R}$ if and only if $\forall\, \epsilon > 0$, $\exists N \in \mathbb{N}$, such that for any $m, n > N$,

$$|f_n(a) - f_m(a)| \le \epsilon.$$

Hence we may write the set in the following form.

$$A = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \{a : |f_i(a) - f_j(a)| \le n^{-1},\ \forall\, i, j > k\} = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} A_{n,k}$$

Suppose $a_m \in A_{n,k}$ and $a_m \to a \in \mathbb{R}$. We have for all $i, j > k$,

$$|f_i(a_m) - f_j(a_m)| \le \frac{1}{n}.$$

For such fixed choice of i, j, we may let $m \to \infty$ yielding that $|f_i(a) - f_j(a)| \le \frac{1}{n}$ meaning that $a \in A_{n,k}$. □

Theorem 0.2 (Heine-Borel theorem)**.** *If F is closed and bounded set in $\mathbb{R}$, then any open cover of F has a finite subcover.*

1 Outer Measure

Question 1.1. *: Can we find a σ-algebra M and a "measure" function $m : M \to [0, +\infty)$ such that $m(\emptyset) = 0$ and $m(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$ for any disjoint union $E_i \in M$?*

Definition 1.2 (Outer measure)**.** *For a set A, we define outer measure to be*

$$m^*(A) = \inf\{\sum_{i=1}^{\infty} |I_i| : A \subset \cup_{i=1}^{\infty} I_i,\ I_i = (a_i, b_i),\ a_i, b_i \in \mathbb{R}\}.$$

For instance, $m^*(I) = |I|$ if I is an bounded interval on real line. (See textbook or do it as exercise)

Example 1.3. *Countable set is of zero outer measure.*

Proof. For any $\epsilon > 0$,

$$\{q_i\}_{i\in\mathbb{N}} \subset \cup_{i=1}^{\infty} (q_i - \frac{\epsilon}{2^i}, q_i + \frac{\epsilon}{2^i})$$

where right hand side is of outer measure smaller than 2ϵ which is arbitrarily small. □

Example 1.4. *A Bounded set E can be approximated by G_δ set.*

Proof. For any $n \in \mathbb{N}$, there exists union of open intervals I_i^n such that $E \subset \cup_{i=1}^{\infty} I_i^n$ and

$$m^*(E) + \frac{1}{n} \geq \sum_{i=1}^{\infty} m^*(I_i^n) \geq m^*(\cup I_i^n) \geq m^*(E)$$

Take $U_n = \cup_{i=1}^{\infty} I_i^n$, $E \subset U_n$ and U_n is a open set. Consider $O = \cap_n U_n$, $E \subset O$ and obeys

$$m^*(E) \leq m^*(O) \leq m^*(U_n) \leq m^*(E) + \frac{1}{n}.$$

Letting $n \to \infty$ yields the result. In particular, O is a G_δ set. □

Now we restrict ourself to those sets which satisfies the ”decomposition”.

Example 1.5. *E is called measurable if for any set A,*

$$m^*(A) = m^*(E \cap A) + m^*(A \setminus E).$$

Theorem 1.6. *All measurable sets form a σ-algebra which contains the Borel sets.*

Example 1.7. *If E is of finite measure then for any $\epsilon > 0$, there exists finite disjoint union E_i such that $\cup_{i=1}^{N} E_i = E$ and $m(E_i) < \epsilon$.*

Proof. If E is bounded, say $E \subset [-N, N]$. Then

$$E = \cup_{i=-\infty}^{\infty} E \cap [-N + i\epsilon, -N + (i+1)\epsilon).$$

As $[a, b)$ is Borel set, $E \cap [-N + i\epsilon, -N + (i+1)\epsilon)$ is measurable and its measure can only smaller than ϵ.

If E is unbounded, $E = \cup_{n \in \mathbb{Z}} E_n$ where $E_n = E \cap [n, n+1)$. E_n are all disjoint. By measurability,

$$m(E) = \sum_{n=-\infty}^{\infty} m(E_n) < +\infty.$$

Hence, $\exists N >> 1$ such that for all $|n| \geq N$,

$$m(\cup_{|n| \geq N} E_n) \leq \sum_{|n| \geq N} m(E_n) < \epsilon.$$

So $E = E_1 \cup E_2$ where $E_1 = \cup_{|n| \geq N} E_n$ and $E_2 = \cup_{|n| < N} E_n$.

E_2 is now bounded, we can repeat the argument of bounded set on E_2 and hence we have the conclusion. □